CONTINUOUS SOLUTIONS TO ALGEBRAIC FORCING EQUATIONS

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Abstract. We ask for a given system of polynomials $f_1, \ldots, f_n$ and f over the complex numbers $\mathbb{C}$ when there exist continuous functions $q_1, \ldots, q_n$ such that $q_1 f_1 + \ldots + q_n f_n = f$. This condition defines the continuous closure of an ideal. We give inclusion criteria and exclusion results for this closure in terms of the algebraically defined axes closure. Conjecturally, continuous and axes closure are the same, and we prove this in the monomial case.

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Introduction

Let $R = \mathbb{C}[z_1, \ldots, z_m]$ be the polynomial ring over the complex numbers. Every polynomial $f \in R$ defines a (holomorphic and) continuous function $\mathbb{C}^m \longrightarrow \mathbb{C}$. Let $I = (f_1, \ldots, f_n)$ be an ideal and $f \in I$. The containment $f \in I$ means that there exist polynomials $g_1, \ldots, g_n \in R$ such that $f = g_1 f_1 + \ldots + g_n f_n$ holds. If we allow $g_1, \ldots, g_n$ to be holomorphic functions or formal power series, then we cannot express additional polynomials as such a linear combination. On the other hand, if we allow arbitrary functions $\mathbb{C}^m \longrightarrow \mathbb{C}$, then it is easy to see that every polynomial inside the radical $\operatorname{rad}(I)$ can be expressed in this way.

In this paper we address the question under which condition we can write $f = q_1 f_1 + \ldots + q_n f_n$ with continuous functions $q_i : \mathbb{C}^m \longrightarrow \mathbb{C}$. Putting it another way: when does the so-called *forcing equation* $f_1 T_1 + \ldots + f_n T_n = f$ has a continuous solution? For a given ideal, the set of $f \in R$ such that there exists a continuous solution form an ideal, which we denote by I^{cont} and which we call the *continuous closure* of I (Section 1). For a principal ideal $I = (g)$ we always have $I = I^{\text{cont}}$, since a rational function f/g has a continuous extension to $\mathbb{C}^m$ only if f is a multiple of g in R. The easiest example for $I \neq I^{\text{cont}}$ is given by $z^2 w^2 \notin (z^3, w^3)$, but $z^2 w^2 \in (z^3, w^3)^{\text{cont}}$ (we will also see that $zw \notin (z^2, w^2)^{\text{cont}}$).

Our main question is whether there exists an algebraic description of the continuous closure. An easier task is to give good algebraically defined approximations. For this we introduce the *axes closure* (Section 4), which is defined in the following way: an element f in a commutative ring R belongs to the (K-)axes closure I^{ax} if and only if for every ring homomorphism $\varphi : R \longrightarrow T$, where $\operatorname{Spec} T$ is a *scheme of axes* over K, $\varphi(f)$ belongs to the extended ideal $\varphi(I)T$. A scheme of axes over K is given by a one dimensional K-algebra of finite type with normal components

meeting in one closed point P such that the completion (at the meeting point) is isomorphic to $\kappa(P)[[x_1, \ldots, x_k]]/(x_i x_j, i \neq j)$ (Section 3).

The basic observation is that for such rings over $K = \mathbb{C}$ the identity $I = I^{\rm cont}$ holds (Lemma 3.5), therefore they serve as a category of testrings. It follows that $I^{\rm cont} \subseteq I^{\rm ax}$ (Corollary 4.7), which implies strong restrictions, since $I^{\rm ax}$ itself is inside the weak subintegral closure (see [10], [8]) and in particular inside the integral closure of the ideal. For example, $zw \notin (z^2, w^2)^{\rm ax}$, since restricted to the cross given by $(z+w)(z-w) = 0$ the element does not belong to the extended ideal (on the two branches there exist solutions, but they do not fit together), and hence $zw \notin (z^2, w^2)^{\rm cont}$ (but $zw \in \overline{(z^2, w^2)}$, its integral closure). We conjecture that $I^{\rm cont} = I^{\rm ax}$, and we prove this in the case of a monomial ideal.

To obtain inclusion results for the continuous closure one has to construct continuous functions. Suppose that $I = (f_1, \ldots, f_n)$ is an ideal primary to the maximal ideal $(z_1, \ldots, z_m)$ and $f \in R$. In this case we can write $f = \sum_{i=1}^n \frac{f \overline{f}_i}{|f_1|^2 + \ldots + |f_n|^2} f_i$. Here $\frac{f \overline{f}_i}{|f_1|^2 + \ldots + |f_n|^2}$ is a continuous function on $\mathbb{C}^m - \{0\}$, and the question is whether we can extend this function continuously to the whole $\mathbb{C}^m$. This can be done in the homogeneous situation under certain degree conditions (Theorem 1.3). We also show that $f \in I^{\rm cont}$ under the condition that there exist numbers $k < d$ such that $f^k \in I^d$ (Theorem 2.3, Corollary 4.4).

For a monomial ideal $I = (z^\gamma, \gamma \in \Gamma)$, $\Gamma \subseteq \mathbb{N}^m$, it follows that the monomials z^τ with exponent τ in the interior of the convex hull $\overline{\Gamma}$ of Γ in $\mathbb{R}^m_+$ belong to the continuous closure (Theorem 6.1). The same inclusion result holds for the axes closure over an arbitrary field (Theorem 5.1). For a monomial z^τ with exponent $\tau \in \overline{\Gamma} - \overline{\Gamma}^\circ$ on the border we will show that $z^\tau \notin I^{\rm ax}$ (unless $z^\tau \in I$) by reducing to the case where I is generated by all monomials of degree d with the exception of one monomial z^τ (Corollary 5.3). These results put together prove for a monomial ideal that $I^{\rm cont} = I^{\rm ax}$ is the monomial ideal $(z^\gamma, \gamma \in \Gamma \cup \overline{\Gamma}^\circ))$ (Theorem 5.1 and Theorem 6.1).

This work started in looking for a test category of rings for the weak subintegral closure ([10],[3]) as presented by M. Vitulli at the conference on 'Valuation Theory and Integral Closures in Commutative Algebra', Ottawa, Canada, July 2006, so I would like to thank the organizers of this conference. I thank T. Gaffney and M. Vitulli for discussions during the conference and after. Furthermore I thank V. Bavula, D. Gepner, M. Hochster, N. Jarvis, A. Kaid, M. Katzman, J. Manoharmayum, R. Sanchez and R. Sharp for their interest and helpful remarks.

1. The continuous closure

Definition 1.1. Let R be a finitely generated $\mathbb{C}$-algebra, $X = \operatorname{Spec} R$ and $X(\mathbb{C})$ the corresponding complex space with the complex topology. Let $I = (f_1, \ldots, f_n)$ be an ideal in R and $f \in R$. We say that $f \in I^{\rm cont}$, the *continuous closure* of I, if there exist continuous functions $q_i : X(\mathbb{C}) \longrightarrow \mathbb{C}$ such that $f = q_1 f_1 + \ldots + q_n f_n$ as continuous functions on $X(\mathbb{C})$.

Another way to think of the continuous closure is to look at the ring homomorphism $R \longrightarrow C_{\mathbb{C}}(X(\mathbb{C}))$ (which is an inclusion if R is reduced), where we denote by $C_{\mathbb{C}}(X)$ the ring of continuous complex-valued functions on a topological space X. Then I^{cont} is the contraction of the extension of the ideal I. In particular, I^{cont} does not depend on the choice of ideal generators.

Remark 1.2. It is also sometimes helpful to see the continuous closure in the context of forcing algebras. This is the R-algebra $B = R[T_1, \ldots, T_n]/(f_1T_1 + \ldots + f_nT_n + f)$. Set $Y = \operatorname{Spec} B$. Then $f \in I^{\text{cont}}$ if and only if the natural projection $Y(\mathbb{C}) \longrightarrow X(\mathbb{C})$ has a continuous section $s : X(\mathbb{C}) \longrightarrow Y(\mathbb{C})$. This viewpoint allows us immediately to extend the notion of continuous closure of an ideal to the continuous closure of an R-submodule $N \subseteq M$ inside a finitely generated R-module M. If $R^n \xrightarrow{D} R^m \longrightarrow M/N \longrightarrow 0$ is a representation, where $D = (f_{ij})_{ij}$ is a $m \times n$-matrix with entries in R, and $\tilde{s} = (s_1, \ldots, s_m) \in R^m$ denotes a representation of $s \in M$, then $s \in N^{\text{cont}}$ if and only if the matrix equation $D(q_1, \ldots, q_n)^t = (s_1, \ldots, s_m)^t$ has a continuous solution. However, we will not pursuit this generalization in this paper. Neither will we be concerned with the Grothendieck topology which arises by taking morphisms of finite type which allow a continuous section as covers (see [2]).

Our first inclusion result for the continuous closure is given by the following theorem.

Theorem 1.3. *Let R be a finitely generated standard-graded $\mathbb{C}$-algebra and $X = \operatorname{Spec} R$, $X(\mathbb{C})$ the corresponding complex analytic space. Let $I \subseteq R$ be a homogeneous R_+-primary ideal with homogeneous ideal generators $f_1, \ldots, f_n$ of degree $d_1, \ldots, d_n$. Let f be a homogeneous element of degree $d > \max d_i$. Then there exist continuous complex-valued functions $q_i : X(\mathbb{C}) \longrightarrow \mathbb{C}$ such that $f = q_1f_1 + \ldots + q_nf_n$ or in other words $f \in I^{\text{cont}}$.*

Proof. Since the only common zero of the f_i is the vertex (the origin) $0 \in X$, they generate the unit ideal in $C_{\mathbb{C}}(X(\mathbb{C}) - \{0\})$ (see the following Remark 1.4). So there exist continuous functions $\varphi_i : X(\mathbb{C}) - \{0\} \longrightarrow \mathbb{C}$ such that $f(z) = \varphi_1(z)f_1(z) + \ldots + \varphi_n(z)f_n(z)$ for all $z \in X(\mathbb{C}) - \{0\}$.

Consider a closed homogeneous embedding $X \subseteq \mathbb{A}^k_{\mathbb{C}}$ and $X(\mathbb{C}) \subseteq \mathbb{C}^k$. Set $S = X(\mathbb{C}) \cap S^{2k-1}$, where $S^{2k-1} = \{z \in \mathbb{C}^k : |z| = 1\}$ is the real sphere. Note that S is a compact closed subset of $X(\mathbb{C})$. We restrict φ_i, $i = 1, \ldots, n$, to S to get continuous functions $\varphi_i : S \longrightarrow \mathbb{C}$. The idea is to extend these functions homogeneously to $X(\mathbb{C}) - \{0\}$ and to use the degree condition to show that it is even extendable to the vertex.

Let $h : X(\mathbb{C}) - \{0\} \longrightarrow S$ be the continuous function given by $h(z) = z/|z|$. Then $z = |z|h(z)$. Since we are on a homogeneous affine variety, if $z \in X(\mathbb{C})$, then also $tz \in X(\mathbb{C})$ for $t \in \mathbb{C}$ (we need it only for $t \in \mathbb{R}_+$). A homogeneous function f on $X(\mathbb{C})$ of degree d has the property that $f(tz) = t^d f(z)$ for $t \in \mathbb{C}^\times$. So in particular we have $f(z) = f(|z|\frac{z}{|z|}) = |z|^d f(h(z))$ for $z \neq 0$.

Define for $z \in X(\mathbb{C}) - \{0\}$ the functions $q_i(z) = |z|^{d-d_i}\varphi_i(h(z))$. Since S is compact, the continuous functions φ_i are bounded on S and since $d > d_i$ we have

$\lim_{|z| \longrightarrow 0} |z|^{d-d_i} \varphi_i(h(z)) = 0$ (in particular, this limit exists). So these functions extend to continuous functions on $X(\mathbb{C})$. We get for $z \neq 0$ the equations

$$\begin{aligned} f(z) &= |z|^d f(h(z)) \\ &= |z|^d \Big(\varphi_1(h(z)) f_1(h(z)) + \ldots + \varphi_n(h(z)) f_n(h(z)) \Big) \\ &= |z|^d \Big(|z|^{d_1 - d} q_1(z) |z|^{-d_1} f_1(z) + \ldots + |z|^{d_n - d} q_n(z) |z|^{-d_n} f_n(z) \Big) \\ &= q_1(z) f_1(z) + \ldots + q_n(z) f_n(z) \, . \end{aligned}$$

By continuity, this equation holds also in the vertex. □

Remark 1.4. We recall the argument that a family of complex-valued functions $f_1, \ldots, f_n$ on a topological space X without common zeros generate the unit ideal in $C_{\mathbb{C}}(X)$. With f_i also its complex conjugate $\overline{f_i}$ is continuous. Hence $g = \sum_{j=1}^n \overline{f_j} f_j$ belongs to the ideal $I = (f_1, \ldots, f_n) C_{\mathbb{C}}(X)$. Since $g = \sum_{j=1}^n \overline{f_j} f_j = \sum_{j=1}^n |f_j|^2 \geq 0$, the set of zeros of this real valued function is the common zero locus of the complex-valued functions $f_1, \ldots, f_n$. So if the f_i have no common zero, then g has no zero at all and $1/g$ is a continuous function on X. Hence g is a unit.

Therefore, in the situation of a primary ideal $I = (f_1, \ldots, f_n)$ in a standard-graded $\mathbb{C}$-algebra R and $f \in R$, $X = \operatorname{Spec} R$, we get on $X(\mathbb{C}) - \{0\}$ the continuous coefficient functions

$$\varphi_i = \frac{f \overline{f}_i}{f_1 \overline{f}_1 + \ldots + f_n \overline{f}_n} = \frac{f \overline{f}_i}{|f_1|^2 + \ldots + |f_n|^2} \, .$$

If the f_i are homogeneous of the same degree, then the real-homogeneous extension of the restriction to S used in the proof of Theorem 1.3 gives us these functions back. In general, if all these φ_i have a limit in the origin, then they extend to continuous functions on $X(\mathbb{C})$ and hence $f \in (f_1, \ldots, f_n)^{\text{cont}}$. So these are natural candidates to look at for continuous solutions. However, even if $f \in (f_1, \ldots, f_n)^{\text{cont}}$ these functions do not always have a continuous extension. For $I = (z, w)$ and $f = z$ we get $\varphi_1 = \frac{z \overline{z}}{|z|^2 + |w|^2} = \frac{|z|^2}{|z|^2 + |w|^2}$, which does not have a limit in the origin (look at the limits for $z = 0$ and $w = 0$). The function is however bounded around the origin, and in fact one can deduce from a characterization of the integral closure due to Teissier that f belongs to the integral closure of $f_1, \ldots, f_n$ if and only if one can write $f = q_1 f_1 + \ldots + q_n f_n$ with locally bounded functions q_i; see [1].

Corollary 1.5. *Let $f \in \mathbb{C}[z_1, \ldots, z_m]$ be a homogeneous polynomial of degree $d \geq 1$. Then there exist continuous functions $q_i : \mathbb{C}^m \longrightarrow \mathbb{C}$ such that $f = q_1 z_1^{d-1} + \ldots + q_m z_m^{d-1}$.*

Proof. This follows directly from Theorem 1.3. □

Example 1.6. Let $I = (z_1^{d_1}, \ldots, z_m^{d_m})$ and $f \in \mathbb{C}[z_1, \ldots, z_m]$ of degree $d > \max d_i$. Then on $\mathbb{C}^m - \{0\}$ we write

$$f(z) = \Big(\frac{f(z) \overline{z}_1^{d_1}}{\sum_{j=1}^m |z_j|^{2d_j}}\Big) z_1^{d_1} + \ldots + \Big(\frac{f(z) \overline{z}_m^{d_m}}{\sum_{j=1}^m |z_j|^{2d_j}}\Big) z_m^{d_m} \, .$$

So in the proof of Theorem 1.3 we have $\varphi_i(z) = \frac{f(z)\overline{z}_i^{d_i}}{\sum_{j=1}^m |z_j|^{2d_j}}$ and accordingly $q_i(z) = |z|^{d-d_i}\varphi_i(\frac{z}{|z|}) = |z|^{d-d_i}\frac{f(\frac{z}{|z|})(\frac{\overline{z}_i}{|z|})^{d_i}}{\sum_{j=1}^m |\frac{z_j}{|z|}|^{2d_j}}$. If the $d_i = e$ are constant, then this is just $q_i(z) = \frac{f(z)\overline{z}_i^e}{\sum_{j=1}^m |z_j|^{2e}}$. If $I = (z^e, w^e)$ and $f = z^r w^s$ is a monomial, $r, s < e$, then $q_1(z, w) = \frac{z^r w^s \overline{z}^e}{|z|^{2e}+|w|^{2e}} = \frac{|z|^{2e} w^s \overline{z}^{e-r}}{|z|^{2e}+|w|^{2e}}$ and $q_2(z, w) = \frac{|w|^{2e} z^r \overline{w}^{e-s}}{|z|^{2e}+|w|^{2e}}$.

Remark 1.7. We give an alternative construction for continuous functions in the situation mentioned in the last example, so $I = (z^e, w^e) \subset \mathbb{C}[z, w]$ and $f = z^r w^s$ with $r, s < e$ and $r + s > e$.

Consider the projective line $\mathbb{P}^1_{\mathbb{C}} \cong S^2$ and denote its complex variable by v, hence $v \in \mathbb{C}$ or $v = \infty$. Consider the function $\psi(v) = \frac{1}{v^e} - \frac{v^s}{v^e}$ in a (complex) neighborhood of $v = \infty$. It is a continuous function in such a neighborhood (and even an algebraic function on $\mathbb{P}^1_{\mathbb{C}} - \{0\}$). We can extend it as a continuous function to the whole sphere, and denote the resulting function again by $\psi(v)$. Note that the function $v^e\psi(v) + v^s$ is also a continuous function on S^2. This is true in a neighborhood of ∞, because the function is even constant $= 1$ there, by the definition of ψ and for $v \neq \infty$ it is clear anyway.

Now we use the substitution $v = w/z$. We set for $(z, w) \neq (0, 0)$

$$q_2(z, w) = z^{r+s-e}\psi(\frac{w}{z}) .$$

As ψ is a bounded (since continuous) function on S^2 and $r + s > e$ this function is continuous on $\mathbb{C}^2 - \{0\}$ and extends continuously to $\mathbb{C}^2$ with value 0 at the origin. We set

$$q_1(z, w) = -z^{r+s-e}((\frac{w}{z})^e\psi(\frac{w}{z}) + (\frac{w}{z})^s) .$$

Again, this gives a continuous function on $\mathbb{C}^2$. We check for $z \neq 0$ (this we need for resolving the bracket)

$$\begin{aligned} z^e q_1 + w^e q_2 + z^r w^s &= -z^e z^{r+s-e}\Big((\frac{w}{z})^e\psi(\frac{w}{z}) + (\frac{w}{z})^s\Big) + w^e z^{r+s-e}\psi(\frac{w}{z}) + z^r w^s \\ &= -z^{r+s-e} w^e \psi(\frac{w}{z}) - z^r w^s + w^e z^{r+s-e}\psi(\frac{w}{z}) + z^r w^s \\ &= 0 . \end{aligned}$$

For $z = 0$ we have $q_2(0, w) = 0$, again since $r + s > e$. Hence also under this condition the equation holds.

2. Powers of functions and powers of ideals

We want to show that $f^k \in I^d$ for $k < d$ implies that $f \in I^{\text{cont}}$. For this we first prove two lemmas.

Lemma 2.1. *The continuous closure is persistent, that is if $\varphi : R \longrightarrow S$ is a $\mathbb{C}$-algebra homomorphism of finitely generated $\mathbb{C}$-algebras, I is an ideal in R and $f \in R$ with $f \in I^{\text{cont}}$, then also $\varphi(f) \in (IS)^{\text{cont}}$.*

Proof. Let $I = (f_1, \ldots, f_n)$. Then the condition means that $f = q_1 f_1 + \ldots + q_n f_n$ with continuous functions $q_i : (\operatorname{Spec} R)(\mathbb{C}) \longrightarrow \mathbb{C}$. Pulling these functions back via the continuous mapping $\varphi^* : (\operatorname{Spec} S)(\mathbb{C}) \longrightarrow (\operatorname{Spec} R)(\mathbb{C})$ gives $\varphi(f) = (q_1 \circ \varphi^*)\varphi(f_1) + \ldots + (q_n \circ \varphi^*)\varphi(f_n)$. This shows that $\varphi(f) \in (\varphi(f_1), \ldots, \varphi(f_n))^{\mathrm{cont}} = (IS)^{\mathrm{cont}}$. $\square$

The following Lemma deals with the generic situation of the proposed question.

Lemma 2.2. *Let $k < d$ and let*

$$R = \mathbb{C}[w, t_\nu, |\nu| = d, z_1, \ldots, z_n]/(w^k - \sum_{\nu, |\nu|=d} t_\nu z^\nu)\,.$$

Then $w \in (z_1, \ldots, z_n)^{\mathrm{cont}}$ in R.

Proof. According to Remark 1.4 we look at the functions $\frac{w\overline{z}_i}{|z_1|^2+\ldots+|z_n|^2}$ defined on $D(z_1, \ldots, z_n) \subset X(\mathbb{C})$ ($X = \operatorname{Spec} R$). We want to show that they have limit 0 as a point converges to $V(z_1, \ldots, z_n)$ and hence that they have a continuous extension from $D(z_1, \ldots, z_n)$ to $X(\mathbb{C})$. The modulus of such a function is $\frac{|w||z_i|}{|z_1|^2+\ldots+|z_n|^2} = \frac{|w|}{\sqrt{|z_1|^2+\ldots+|z_n|^2}}\frac{|z_i|}{\sqrt{|z_1|^2+\ldots+|z_n|^2}}$ and the second factor is bounded by 1, so we only have to deal with the first one. The function $\frac{|w|}{\sqrt{|z_1|^2+\ldots+|z_n|^2}} = \frac{|w|}{|(z_1,\ldots,z_n)|}$ has limit 0 for a sequence $P_m \in D(z_1, \ldots, z_n)$ converging to $P \in V(z_1, \ldots, z_n)$ if and only if this is true for a (natural) power of it. Hence we replace this function by $\frac{|w|^k}{|(z_1,\ldots,z_n)|^k} = \frac{|\sum_\nu t_\nu z^\nu|}{|(z_1,\ldots,z_n)|^k}$. We may look at the summands separately. Since P_m converges, the values of $t_\nu(P_m)$ are bounded, and so we consider $\frac{|z^\nu|}{|(z_1,\ldots,z_n)|^k} = \frac{|z_1^{\nu_1}\cdots z_n^{\nu_n}|}{|(z_1,\ldots,z_n)|^k}$, where $|\nu| = \nu_1 + \ldots + \nu_n = d > k$. We can write $\frac{|z_1^{\nu_1}\cdots z_n^{\nu_n}|}{|(z_1,\ldots,z_n)|^k} =$

$$\frac{|z_1^{\nu_1}|\cdots|z_n^{\nu_n}|}{\sqrt{|z_1|^2+\ldots+|z_n|^2}^k} = z^\mu \Big(\frac{|z_1|}{\sqrt{|z_1|^2+\ldots+|z_n|^2}}\Big)^{k_1} \cdots \Big(\frac{|z_n|}{\sqrt{|z_1|^2+\ldots+|z_n|^2}}\Big)^{k_n},$$

where $|\mu| = d - k > 0$. As $P_m \longrightarrow P$, $z_i \longrightarrow 0$ and so this function goes to 0. $\square$

Theorem 2.3. *Let R be a finitely generated $\mathbb{C}$-algebra, let I be an ideal and $f \in R$. Suppose that there exist numbers $k < d$ such that $f^k \in I^d$. Then $f \in I^{\mathrm{cont}}$.*

Proof. Let $I = (f_1, \ldots, f_n)$ and write the condition as

$$f^k = \sum_{\nu, |\nu|=d} g_\nu f_1^{\nu_1} \cdots f_n^{\nu_n},\ g_\nu \in R\,.$$

We have a ring homomorphism from the generic situation to this special situation, namely

$$\mathbb{C}[w, t_\nu, |\nu| = d, z_1, \ldots, z_n]/(w^k - \sum_{\nu, |\nu|=d} t_\nu z^\nu) \longrightarrow R,\ z_i \mapsto f_i,\ t_\nu \mapsto g_\nu,\ w \mapsto f\,.$$

Lemma 2.2 shows that $w \in (z_1, \ldots, z_n)^{\mathrm{cont}}$ in the ring on the left hand side. By the persistence of the continuous closure (Lemma 2.1) $f \in I^{\mathrm{cont}}$ follows. $\square$

Remark 2.4. The condition in Theorem 2.3 can not be weakened. Example 5.5 below shows that the condition $f^k \in I^k$ for all $k \geq 2$ does not imply $f \in I^{\text{cont}}$. We will show in Corollary 4.4 that under the assumption of Theorem 2.3 the element f belongs also to the axes-closure.

Theorem 2.3 has also a purely topological analog. If X is a topological space an f and $f_1, \ldots, f_n$ are continuous complex-valued functions on X such that $f^k \in (f_1, \ldots, f_n)^d$ (in $C_{\mathbb{C}}(X)$) for some $k < d$, then already $f \in (f_1, \ldots, f_n)$. We can use an equation $f^k = \sum_{\nu, |\nu| = d} g_\nu f_1^{\nu_1} \cdots f_n^{\nu_n}$ with $g_\nu \in C_{\mathbb{C}}(X)$ to get a continuous mapping $(f_1, \ldots, f_n, g_\nu, f) : X \longrightarrow V = V(w^k - \sum_{\nu, |\nu|=d} t_\nu z^\nu) \subseteq \mathbb{C}^\ell$. Since $w \in (z_1, \ldots, z_n) C_{\mathbb{C}}(V)$ by Lemma 2.2, it follows $f \in (f_1, \ldots, f_n) C_{\mathbb{C}}(X)$.

3. K-schemes of axes

We want to control the continuous closure algebraically, in particular we want to find exclusion criteria for it. We fix a field K and work in the category of finitely generated K-algebras. Whenever we talk about the continuous closure we suppose that $K = \mathbb{C}$.

We have immediately the inclusions $I \subseteq I^{\text{cont}} \subseteq \text{rad}(I)$. In fact $f \in \text{rad}(I)$ holds if and only if $f = g_1 f_1 + \ldots + g_n f_n$ with arbitrary functions g_i. A better algebraic approximation is given by the integral closure. One of the equivalent characterizations of the integral closure $\overline{I}$ of an ideal I in a K-algebra R of finite type is that for every normal affine curve $C = \operatorname{Spec} T$ over K and every K-morphism $\varphi : C \longrightarrow \operatorname{Spec} R$ we have $\varphi(f) \in \varphi(I)T$ [1]. If $K = \mathbb{C}$ and $f \in I^{\text{cont}}$, then for every such curve it follows (by the persistence of the continuous closure, Lemma 2.1) that $\varphi(f) \in (\varphi(I)T)^{\text{cont}}$. But for a normal (=regular) curve always $I = I^{\text{cont}}$ holds, since I is locally a principal ideal.

However, also the inclusion $I^{\text{cont}} \subseteq \overline{I}$ is strict. For example, $zw \in \overline{(z^2, w^2)}$, but zw does not belong to the continuous closure, as we will see (Corollary 5.3). Also, the weak subintegral closure ([10], [8]) is too big, as $zw^2 \in (z^3, z^2 w, w^3)^{\text{wsi}}$, but again not in the continuous closure (Example 5.5). So our goal here is to find a reasonably big category of one dimensional K -schemes where the continuous closure is the identical closure and which will be later on a test category for exclusion results.

Definition 3.1. A *scheme of axes* over K is a one-dimensional reduced affine scheme $C = \operatorname{Spec} T$ of finite type such that its integral components are normal, meet transversally in one point P (the origin) and that the embedding dimension at P equals the number of components. We call T a *ring of axes*.

Remark 3.2. An equivalent characterization is that the completion of the local ring at the origin P is isomorphic to $\kappa(P)[[x_1, \ldots, x_k]]/(x_i x_j, i \neq j)$. A scheme of axes is a *seminormal* one-dimensional scheme and every seminormal one-dimensional local scheme of finite type over an algebraically closed field looks after completion like this (J. Tong called these singularities in a talk in Münster, June 2006, 'singularities of coordinate axe type'; for the notion of seminormal rings see [6], [7], [3]). The easiest example is given by the standard scheme of axes $K[x_1, \ldots, x_k]/(x_i x_j, i \neq j)$.

The rings $K[x,y]/(xy(x+y))$ and $K[x,y]/(y(y+x^2))$ do not give scheme of axes, though their components are smooth and though they give for $K=\mathbb{C}$ topologically a space of axes.

The property of being a scheme of axes depends on the field K. The ring $T=\mathbb{R}[z,w]/(z^2+w^2)$ is seminormal, its normalization is $\mathbb{C}[z]$ (with a non-trivial extension of residue class fields). As $\operatorname{Spec} T$ is integral with one non-normal component, it is not a scheme of axes. The tensoration $T\otimes_{\mathbb{R}}\mathbb{C}=\mathbb{C}[z,w]/((z+iw)(z-iw))$ is however a ring of axes. If T is a K-algebra such that $T\otimes_K \overline{K}$ is a ring of axes, then there exist also a finite extension $K\subseteq L\subseteq \overline{K}$ such that $T\otimes_K L$ is a ring of axes over L and hence over K.

If C_j are smooth affine curves over K with fixed K-points $P_j\in C_j$, $j=1,\ldots,k$, then

$$C=\{(z_1,\ldots,z_k)\in C_1\times\ldots\times C_k : \exists j \text{ such that } z_i=P_i \text{ for all } i\neq j\}$$

is a scheme of axes with origin $P=P_1\times\ldots\times P_k$ and components $C_j\cong P_1\times\ldots\times P_{j-1}\times C_j\times P_{j+1}\times\ldots\times P_k$.

Let $C=\operatorname{Spec} T$ be a scheme of axes with axes (integral components) L_j, $j=1,\ldots,k$. After localizing at P, each L_j is the spectrum of a discrete valuation domain. In particular, for every j there exists a valuation val_j on T, which assigns to $f\in T$ the order $\mathrm{val}_j(f)=\mathrm{val}_j(f|L_j)$ (which is ∞ if $f|L_j=0$). For an ideal I we set $\mu_j=\mathrm{val}_j(I)=\min\{\mathrm{val}_j(f): f\in I\}$. We describe the ideals in a complete scheme of axes.

Lemma 3.3. *Let K be a field and let $T=K[[x_1,\ldots,x_k]]/(x_ix_j, i\neq j)$ or $T=\mathbb{C}\{\{x_1,\ldots,x_k\}\}/(x_ix_j, i\neq j)$, the ring of convergent power series (convergent in some open complex neighborhood of the origin). Let $I\subseteq T$ be an ideal. Let μ_j be the order (possibly ∞) of the ideal on the jth axis given by $x_i=0$ for $i\neq j$. Then $(x_j^{\mu_j+1})\subseteq I$ for the indices j such that μ_j is finite. Moreover, the ideal I has after reindexing a system of generators of the form*

$$f_i=x_i^{\mu_i}+\lambda_{i,n+1}x_{n+1}^{\mu_{n+1}}+\ldots+\lambda_{i,\ell}x_\ell^{\mu_\ell},\ i=1,\ldots,n\,.$$

Here $\lambda_{i,j}\in K$ and x_j, $1\leq j\leq \ell$, correspond to the components where μ_j is finite.

Proof. The statement is true for $I=(1)$, so assume that $I\neq(1)$. Every non unit $f\in T$ can be written as $f=\sum_{j=1}^k x_j^{\delta_j}G_j(x_j)$, where $\delta_j=\mathrm{val}_j(f)\geq 1$ for $f|_{L_j}\neq 0$ and where $G_j=0$ or G_j is a (convergent) power series in x_j with non zero constant term (and a unit in T). If the order of I on the jth component is $\mu_j<\infty$, then there exists an element $f\in I$ with x_j-exponent $\delta_j=\mu_j$. Then $x_jf=x_j^{\mu_j+1}G_j(x_j)$, and so $x_j^{\mu_j+1}\in I$.

Assume after reindexing that the variables $x_1,\ldots,x_\ell$ occur with finite order in I (so the other variables do not occur at all). We look at a system of generators for I which encompasses the $x_j^{\mu_j+1}$, $j=1,\ldots,\ell$. Then the other generators can be written as $g=\sum_{j=1}^{\ell}\lambda_j(g)x_j^{\mu_j}$ with coefficients $\lambda_j(g)\in K$. Applying Gauss elimination and

reindexing gives a generating system as stated, where also the $x_j^{\mu_j+1}$ are not necessary anymore. $\square$

Corollary 3.4. *Let $T = (K[x_1, \dots, x_k]/(x_i x_j, i \neq j))_{(x_1,\dots,x_k)}$ or its completion $T = K[[x_1, \dots, x_k]]/(x_i x_j, i \neq j)$ and let I be an ideal, $f \in T$. Let val_j be the valuation on the jth axes given by $x_i = 0, i \neq j$. If $\delta_j = \operatorname{val}_j(f) > \mu_j(I) = \mu_j$ for all j, then $f \in I$.*

Proof. We only deal with the complete case. By Lemma 3.3, $x_j^{\mu_j+1} \in I$. We can write $f = x_1^{\delta_1} G_1 + \dots + x_k^{\delta_k} G_k$, where $G_j \in T$. The condition means that $\delta_j > \mu_j$, hence $f \in (x_1^{\mu_1+1}, \dots, x_k^{\mu_k+1}) \subseteq I$. $\square$

We want to show now that for $K = \mathbb{C}$ the continuous closure on a scheme of axes is the identical closure operation.

Lemma 3.5. *Let $T = \mathbb{C}\{\{x_1, \dots, x_k\}\}/(x_i x_j, i \neq j)$. Let $I = (f_1, \dots, f_n)$ and $f \in T$ be given. Suppose that there exists a complex open neighborhood U of the origin where f and f_i converge and that there exist continuous complex-valued functions $q_1, \dots, q_n$ defined on U such that $f = q_1 f_1 + \dots + q_n f_n$ (as continuous functions on U). Then $f \in (f_1, \dots, f_n)$ holds in T.*

Proof. Let $I \neq (1)$ and let μ_j be the order of the ideal along the axis L_j. As explained in Lemma 3.3, there is a system of ideal generators looking like

$$f_i = x_i^{\mu_i} + \lambda_{i,n+1} x_{n+1}^{\mu_{n+1}} + \dots + \lambda_{i,\ell} x_\ell^{\mu_\ell}, \ i = 1, \dots, n\,.$$

By the condition of continuity restricted to the axis L_j separately it follows that $\operatorname{val}_j(f) \geq \mu_j$. Hence we can write (after subtracting elements in I) $f = \beta_{n+1} x_{n+1}^{\mu_{n+1}} + \dots + \beta_\ell x_\ell^{\mu_\ell} + p(x_{\ell+1}, \dots, x_k)$. Assume now that the continuous functions q_j do their job, i.e. $\sum_{j=1}^n q_j f_j = f$. It follows immediately that $p(x_{\ell+1}, \dots, x_k) = 0$. Then on the axis L_j, $j \leq n$ (given by $x_i = 0$ for $i \neq j$), we get the condition

$$\sum_{i=1}^n (q_i|_{L_j})(f_i|_{L_j}) = (q_j|_{L_j}) x_j^{\mu_j} = f|_{L_j} = 0\,.$$

It follows that $q_j = 0$ on L_j first outside the origin but by continuity also at the origin. So $q_i(0) = 0$ for all $i = 1, \dots, n$.

Assume now that $f \notin I$ and that $\beta_j \neq 0$, $j \geq n+1$. Then on L_j we get the equation $\sum_{i=1}^n (q_i|_{L_j}) \lambda_{i,j} x_j^{\mu_j} = \beta_j x_j^{\mu_j}$ and so $\sum_{i=1}^n (q_i|_{L_j}) \lambda_{i,j} = \beta_j$. This gives a contradiction at the origin. Hence $\beta_j = 0$ for $j > n$ and so in fact $f = 0$. $\square$

Corollary 3.6. *Let $C = \operatorname{Spec} T$ be a scheme of axes over $\mathbb{C}$, $I \subseteq T$ an ideal. Then $I^{\text{cont}} = I$.*

Proof. Let $f \in T$ and $f \in I^{\text{cont}}$. Then we have to check that $f \in I\mathcal{O}_Q$ for every point $Q \in C$. For $Q \neq P$ this is the (easier) case of just one axis, so we may assume that $Q = P$ is the origin. In a small neighborhood of P, C is isomorphic as a complex space to an open (ball) neighborhood of the origin of the (standard) scheme of axes D in $\mathbb{C}^k$. This is also a homeomorphism, so we may assume that $I = (f_1, \dots, f_n)$

and that f, f_i are holomorphic functions defined on $D_r = \{z \in D : |z| < r\}$. By assumption there exist continuous functions q_i such that $f = q_1 f_1 + \ldots + q_n f_n$ on D_r. By Lemma 3.5 this means that $f \in (f_1, \ldots, f_n)\mathbb{C}\{\{x_1, \ldots, x_k\}\}/(x_i x_j, i \neq j)$. Hence also $f \in (f_1, \ldots, f_n)\mathbb{C}[[x_1, \ldots, x_k]]/(x_i x_j, i \neq j)$. This implies by faithful flatness of the completion that $f \in IT_P$. □

4. The axes closure

Definition 4.1. Let I be an ideal in a commutative ring R, $f \in R$. We say that f belongs to the (K-)*axes closure*, $f \in I^{\text{ax}}$ if for every ring homomorphism $\varphi : R \longrightarrow T$, where T is a K-scheme of axes, we have $\varphi(f) \in \varphi(I)T$.

Remark 4.2. We will mainly be interested in the (K)-axes closure for ideals in the category of finitely generated K-algebras. If the base field is understood we will just talk about the axes closure. Some of the results are independent of the base field, but we will not treat this systematically.

The (K)-axes closure of an ideal in a finitely generated K-algebra is inside the integral closure. The integral closure can be tested by homomorphisms to discrete valuation domains, but in the case of a K-algebra R of finite type also by only looking at morphisms from affine normal K-curves to $\operatorname{Spec} R$. Every such curve is a scheme of (one) axes. In [3] we show that the weak subintegral closure as introduced by Leahy and Vitulli in [10] can be tested by morphisms of scheme of axes with only two components (crosses). Hence also $I^{\text{ax}} \subseteq I^{\text{wsi}}$.

Corollary 4.3. *Let R be a commutative ring, $I \subseteq R$ an ideal and $f \in R$. Suppose that for every discrete valuation domain (V, ν) and every ring homomorphism $\varphi : R \longrightarrow V$ we have $\nu(\varphi(f)) > \nu(\varphi(I))$. Then $f \in I^{\text{ax}}$ (independent of any K).*

Proof. This follows directly from Corollary 3.4 and the definition of the axes closure. □

We can deduce the axes version of Theorem 2.3.

Corollary 4.4. *Let R be a commutative ring, $I \subseteq R$ an ideal and $f \in R$. Suppose that $f^k \in I^d$ for some $k < d$. Then $f \in I^{\text{ax}}$.*

Proof. We want to apply Corollary 4.3 and look at $\varphi : R \longrightarrow V$ to a discrete valuation domain. The containment $\varphi(f)^k \in (\varphi(I))^d$ implies the relationship $k\,\nu(\varphi(f)) \geq d\,\nu(\varphi(I))$. This gives $\nu(\varphi(f)) > \nu(\varphi(I))$ for all valuations ν. □

We provide two further propositions which we will use in the computation of the axes closure of a monomial ideal.

Proposition 4.5. *Let K be an algebraically closed field. Suppose that R is a commutative ring and that $f \in I^{\text{ax}}$. Then $\varphi(f) \in \varphi(I)T$ holds also for every ring homomorphism to a seminormal one dimensional affine ring over K.*

Proof. We may assume immediately that $D = \operatorname{Spec} R$ is an affine seminormal one dimensional scheme of finite type over K, and we have to show that $f \in I$ under the condition that this is true for every morphism $C \longrightarrow D$, where C is a K-scheme of axes. Since the containment $f \in I$ is a local property we may assume that D has only one singular point. We know that the completion at the singular point of D looks like the completion of a ring of axes. Let $D_1, \ldots, D_m$ be the integral components of D (which are seminormal, but not normal in general), and let $C_1, \ldots, C_m$ be the normalizations. On each C_i, let $P_{i,j}$, $j \in J_i$, be the closed points mapping to the origin $P_i \in D_i$. Then the family of pointed curves $P_{i,j} \in C_i$, $j \in J_i$, $i = 1, \ldots, m$, glue together in the sense of Remark 3.2 to get a scheme of axes over K, say C and a morphism $C \longrightarrow D$. This morphism is in the completion at the singular point an isomorphism. $\square$

Lemma 4.6. *Let K be an algebraically closed field and let $\theta : R \longrightarrow S$ be a finite and pure (e.g. faithfully flat) homomorphism of K-algebras. Let $I \subseteq R$ be an ideal and $f \in R$. Suppose that $\theta(f) \in (\theta(I)S)^{\mathrm{ax}}$. Then already $f \in I^{\mathrm{ax}}$.*

Proof. We may assume immediately that $R = T$ is a ring of axes over K. Then S is also a one dimensional K-algebra of finite type, let S^{sn} be its seminormalization. Since $f \in (IS)^{\mathrm{ax}}$ (we denote the image of f in S and in S^{sn} again by f), we know by Proposition 4.5 that $f \in IS^{\mathrm{sn}}$. The morphism $\operatorname{Spec} S^{\mathrm{sn}} \longrightarrow \operatorname{Spec} S$ is a homeomorphism. By the geometric criterion [3] for purety for schemes over schemes of axes it follows that also $T \longrightarrow S^{\mathrm{sn}}$ is pure. Hence $f \in I$ in T. $\square$

In the case $K = \mathbb{C}$ we can easily establish the relation between axes closure and continuous closure.

Corollary 4.7. *Let R be a $\mathbb{C}$-algebra of finite type. Then $I^{\mathrm{cont}} \subseteq I^{\mathrm{ax}}$.*

Proof. Let $f \in I^{\mathrm{cont}}$ and let $\varphi : R \longrightarrow T$ be a homomorphism to a $\mathbb{C}$-algebra of axes. By the persistence of the continuous closure we have $\varphi(f) \in (\varphi(I)T)^{\mathrm{cont}}$. By Corollary 3.6 it follows that $\varphi(f) \in \varphi(I)T$, hence $f \in I^{\mathrm{ax}}$. $\square$

Question 4.8. Is for $\mathbb{C}$-algebras of finite type the continuous closure the same as the axes closure? Is this true for the polynomial ring? A purely topological version of this question is whether a continuous mapping $Y \longrightarrow X$ admits a continuous section under the condition that for every continuous mapping $\psi : C \longrightarrow X$ there exists a lifting $\tilde{\psi} : C \longrightarrow Y$, where C is a space consisting of a finite number of lines (or planes) meeting in one point. Minimal topological requirements to make this a reasonable question are that X and Y are locally compact with only finitely many components.

5. The axes closure of a monomial ideal

We want to show that for a monomial ideal Question 4.8 has a positive answer by computing explicitly how its axes closure and continuous closure look like. Recall that for a monomial ideal $I = (z_1^{\gamma}, \ldots, z_n^{\gamma})$ in $K[z_1, \ldots, z_m]$ it is helpful to consider

the set of exponents $\Gamma = \{\gamma : z^\gamma \in I\}$ as a set of integral points $\Gamma \subseteq \mathbb{N}^m \subset \mathbb{R}_+^m$. For example, the integral closure of a monomial ideal is given by the monomial ideal consisting of z^γ such that γ lies in the convex hull $\overline{\Gamma}$ of Γ inside $\mathbb{R}_+^m$ [4, Exercise 4.23]. Also, the weak subintegral closure of a monomial ideal has a combinatorial description in terms of the exponents [8, Proposition 3.3 and Theorem 4.11].

Theorem 5.1. *Let $R = K[z_1, \ldots, z_m]$ be a polynomial ring and let I be a primary monomial ideal. Then I^{ax} consists of I and of the monomials in the interior of the convex hull $\overline{\Gamma}$, where $\Gamma = \{\gamma : z^\gamma \in I\}$ is the corresponding set of exponents.*

Proof. Suppose that a monomial is in the interior $\overline{\Gamma}^\circ$ of the complex hull. Then for every discrete valuation the monomial has bigger order than the ideal (one can also show that the situation of Corollary 4.4 is fulfilled), so by Corollary 4.3 it belongs to I^{ax}.

So let f be a polynomial consisting of monomials which lie on the border and do not belong to the ideal itself. We may assume by Remark 3.2 that K is algebraically closed. We can enlarge the ideal by adding all monomials which are in the support of this polynomial except one. Hence in particular we may assume that f is a monomial, $f = z^\tau$, $\tau \in \overline{\Gamma} - \overline{\Gamma}^\circ$, $\tau \notin \Gamma$. Suppose that the exponent τ lies on the affine hyperplane (forming a border of Γ) spanned by the exponents $\gamma_1, \ldots, \gamma_m$, $\gamma_j \in \Gamma$. A suitable finite transformation $z_j \mapsto z_j^{\delta_j}$ maps the monomials $z^{\gamma_1}, \ldots, z^{\gamma_m}$ to monomials of constant degree d. All other monomials in Γ are mapped to monomials of degree $\geq d$. By Lemma 4.6, the axes closure is not changed by such a finite free extension.

Hence we may assume by filling up the ideal that the ideal is given by all monomials of fixed degree d with the exception of one monomial. Then Corollary 5.3 gives the result. □

Proposition 5.2. *Let K be a field, $R = K[z_1, \ldots, z_m]$, $d \in \mathbb{N}$. Let $I \subset R$ be an ideal generated by polynomials $f_1, \ldots, f_n$ of degree d and let f be another polynomial of degree d. Then $f \in I^{\text{ax}}$ if and only if $f \in I$.*

Proof. We may immediately assume that K is an infinite field. According to Lemma 5.4, let S be a set of k points in K^m with the property that every polynomial of degree $\leq d$ which vanishes on S must be the zero polynomial. Let the k points P_j have coordinates $\alpha_j = (\alpha_{j1}, \ldots, \alpha_{jm})$, $j = 1, \ldots, k$. We consider the ring homomorphism

$$\varphi : K[z_1, \ldots, z_m] \longrightarrow K[x_1, \ldots, x_k]/(x_i x_j, i \neq j)$$

given by

$$z_r \longmapsto \alpha_{1r} x_1 + \ldots + \alpha_{kr} x_k \, .$$

A polynomial g of degree d is sent to $\varphi(g) = g(\alpha_1)x_1^d + \ldots + g(\alpha_k)x_k^d$, where $g(\alpha_j) = g(\alpha_{j1}, \ldots, \alpha_{jm})$. Hence $\varphi(f) \in (\varphi(f_1), \ldots, \varphi(f_n))$ if and only if there exist $c_i \in K$, $i = 1, \ldots, n$, such that $\varphi(f) = \sum_{i=1}^n c_i \varphi(f_i)$, and this means by looking at each component (each coefficient of x_j^d) that $f(\alpha_j) = \sum_{i=1}^n c_i f_i(\alpha_j)$ for all points α_j, $j = 1, \ldots, k$. This means by the choice of α_j that $f = \sum_{i=1}^n c_i f_i$. □

Corollary 5.3. *Let K be a field, $R = K[z_1, \ldots, z_m]$, $d \in \mathbb{N}$. Let $I \subset R$ be a monomial ideal generated by all monomials of degree d with the exception of $z^\tau = z_1^{\tau_1} \cdots z_m^{\tau_m}$, $\sum_{j=1}^m \tau_j = d$. Then $z^\tau \notin I^{\mathrm{ax}}$.*

Proof. This is a special case of Proposition 5.2 □

Lemma 5.4. *Let K be an infinite field, let $d, m \in \mathbb{N}$. Then there exist k points in K^m such that every polynomial of degree $\leq d$ which vanishes at all these points is the zero polynomial.*

Proof. This can be proved by induction over m, see [9, Satz 54.7]. □

Example 5.5. Even for monomial ideals in a two-dimensional polynomial ring the axes closure (and the continuous closure) is smaller than the weak subintegral closure. The easiest example is the ideal $I = (z^3, z^2w, w^3) \subseteq R = K[z, w]$ and $f = zw^2$. Then by [8, Proposition 3.3 and Example 4.12] we have $zw^2 \in I^{\mathrm{wsi}}$. We have directly $(zw^2)^k \in I^k$ for $k \geq 2$. Consider according to Proposition 5.2 the homomorphism $R \longrightarrow T = K[x_1, x_2, x_3, x_4]/(x_ix_j, i \neq j)$ given by $(\mathrm{char}(K) \neq 2)$

$$z \mapsto g = x_1 + x_2 + x_4 \text{ and } w \mapsto h = x_1 + x_3 + 2x_4 \, .$$

Then in T we have

$$g^3 = x_1^3 + x_2^3 + x_4^3, \;\; g^2h = x_1^3 + 2x_4^3, \;\; h^3 = x_1^3 + x_3^3 + 8x_4^3$$

and $gh^2 = x_1^3 + 4x_4^3$. Looking at the coefficients we see that gh^2 is not in the extended ideal (g^3, g^2h, h^3). Hence $zw^2 \notin I^{\mathrm{ax}}$. It follows also for $K = \mathbb{C}$ that $zw^2 \notin I^{\mathrm{cont}}$.

6. The continuous closure of a monomial ideal

We can now put the previous results together and compute the continuous closure of a monomial ideal.

Theorem 6.1. *Let $R = \mathbb{C}[z_1, \ldots, z_m]$ and $I \subset R$ be a primary monomial ideal given by the monomials z^γ, $\gamma \in \Gamma$. Then $I^{\mathrm{cont}} = I^{\mathrm{ax}}$, and this is the monomial ideal given by the monomials in I and the monomials with exponents in the interior of the convex hull $\overline{\Gamma}$ of Γ in $\mathbb{R}_+^m$.*

Proof. We have shown in Corollary 4.7 that $I^{\mathrm{cont}} \subseteq I^{\mathrm{ax}}$ and in Theorem 5.1 that I^{ax} has this description. So we only have to show that the monomials with exponent in the interior $\overline{\Gamma}^\circ$ are inside the continuous closure. If z^τ is inside the interior, then there exist monomials $z^{\gamma_1}, \ldots, z^{\gamma_m} \in I$ such that the exponent τ lies "above" the affine hyperplane spanned by the exponents $\gamma_1, \ldots, \gamma_m$ of these monomials. Let $\gamma_i = \gamma_{i,j}$. There exist positive numbers $\delta_1, \ldots, \delta_m \in \mathbb{N}$ such that $\sum_{j=1}^m \delta_j \alpha_j = d \in \mathbb{N}$ if and only if $\alpha = (\alpha_1, \ldots, \alpha_m)$ belongs to the affine hyperplane spanned by the monomials. In particular, $\sum_{j=1}^m \delta_j \gamma_{i,j} = d$ for $i = 1, \ldots, m$ and $\sum_{j=1}^m \delta_j \tau_{i,j} > d$. Applying the ring homomorphism $z_j \mapsto z_j^{\delta_j}$ brings our situation to an equivalent situation (by Lemma 6.2) where the generating monomials have the same degree d and the monomial in question has strictly bigger degree. Hence by Theorem 1.3 it

belongs to the continuous closure of the ideal (or one can show after this step that $(z^\tau)^k \in I^d$, $k < d$, and then apply Theorem 2.3). □

Lemma 6.2. *Let $\varphi : \mathbb{C}^m \longrightarrow \mathbb{C}^m$ be given by $(z_1, \ldots, z_m) \longrightarrow (z_1^{\delta_1}, \ldots, z_m^{\delta_m})$. Let $I = (f_1, \ldots, f_n)$ be an ideal in the ring $S = C_{\mathbb{C}}(\mathbb{C}^m)$ of continuous complex-valued functions on $\mathbb{C}^m$. Then for $f \in S$ we have $f \in I$ if and only if $f \circ \varphi \in (I \circ \varphi)S$.*

Proof. The ring extension $S \subseteq S$ given by $f \mapsto f \circ \varphi$ is a direct summand exactly as in the case of a polynomial ring because of the existence of the trace map. The Galois group $G = \mathbb{Z}/\delta_1 \times \ldots \times \mathbb{Z}/\delta_m$ acts on $\mathbb{C}^m$ (and on S) with quotient $\mathbb{C}^m$ (and invariant ring S). An element $q \in S$ yields the invariant element $\sum_{\xi \in G} q\xi$, where $(q\xi)(z) = q(\xi_1 z_1, \ldots, \xi_m z_m)$. Applying this to an equation $f \circ \varphi = q_1(f_1 \circ \varphi) + \ldots + q_n(f_n \circ \varphi)$ gives a similar equation for f and $f_1, \ldots, f_n$. □

Example 6.3. The ideal $I = (z^2, w^5)$ in $\mathbb{C}[z, w]$ and $f = zw^3$ gives an example where f does not fulfill the degree assumption of Theorem 1.3, but it is due to Theorem 6.1 in the continuous closure anyway.

Question 6.4. Is it possible to give an effective criterion for $f \in (f_1, \ldots, f_n)^{\text{cont}}$ (in a polynomial ring)? Is, if the ideal generators form a Gröbner basis, the containment inside the continuous closure just a question of whether the initial term of f belongs (in an inductive sense) to the continuous closure of the initial ideal.

Question 6.5. Is the tight closure of an ideal always inside the continuous closure? This is for normal $\mathbb{C}$-domains of finite type a reasonable question. The continuous closure is often bigger, as already the regular case shows. In the homogeneous parameter case $I = (f_1, \ldots, f_n)$ in dimension ≥ 2, we have by the Theorem of Hara [5, Theorem 6.1] that $f \in I^*$ implies (if $f \notin I$) $\deg(f) \geq \sum_{j=1}^n \deg(f_j)$ and so in particular $\deg(f) > \deg(f_j)$ for all j, hence $f \in I^{\text{cont}}$.

There is no inclusion between the continuous closure and the regular closure. It is known that $z \in (x, y)^{\text{reg}}$ in $R = K[x, y, z]/(z^3 - x^3 - y^3)$. However, $R/(x, (y - z)(y - \zeta_3 z)) \cong K[y, z]/(y - z)(y - \zeta_3 z)$ is a ring of two axes, but z is not a multiple of y in this ring.

Remark 6.6. The results of this paper hold probably also if we replace $C_{\mathbb{C}}(X)$ by $C^\infty_{\mathbb{C}}(X)$, the ring of smooth functions i.e. functions which are differentiable in the real sense of arbitrary order. However, modifications are needed as the continuous solutions in Theorem 1.3 and Theorem 2.3 are not smooth. The functions used in Remark 1.7 can be easily replaced by smooth functions.

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